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Casimir effect in hyperbolic polygons

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Abstract
Using the point splitting regularization method and the trace formula for the spectra of quantum–mechanical systems in hyperbolic polygons which are the fundamental domains of discrete isometry groups acting in the two-dimensional hyperboloid we calculate the Casimir energy for massless scalar fields in hyperbolic polygons. The dependence of the vacuum energy on the number of vertices is established.

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1. Introduction

Signs of the Casimir energy is known to be dependent on the dimension, topology, metric properties of manifolds and the shape of boundaries where fields under consideration vanishes [1]. For example, the vacuum energy in the three-dimensional ball is positive [2] whereas in the original parallel plane configuration it is negative [3]. No general approach is known to investigate the dependence of the Casimir energy on the geometry of boundaries for compact domains. Technical difficulties related to the solving of the Dirichlet problem restrict the number of cavities for which explicit results can be obtained. It seems that only those cavities which have additional symmetries (such as the rotational symmetry for the ball) are treatable. Cavities which are the fundamental domains of discrete isometry groups give us another class of solvable physical system: symmetries of isometry discrete groups can be used to find the spectra of such systems. However, the Euclidean spaces admits only finite number of crystallographic groups and consequently only special cavities with fixed shapes can be treated by the reflection method [4].

Hyperbolic spaces in contrast to the Euclidean ones admit infinite number of different cavities which can be treated by the reflection method [5]. This may give us the possibility to find the analytic expression for the Casimir energy as a function of the boundary configuration. We restrict our consideration on the two-dimensional hyperbolic space H^2. We consider polygons bounded by an arbitrary number geodesics (the analog of polygons in the flat case). Duality between the configuration and momentum spaces are not well known for hyperbolic spaces: no individual formulae for eigenvalues of the Laplace–Beltrami operator on such domains are known. The Selberg's trace formula is currently the only available tool to analyze the sums over the spectra of quantum–mechanical systems on hyperbolic manifolds. This formula manifests the duality between the spectra of compact manifolds and geodesics—elements of discrete hyperbolic groups which are the fundamental groups for these manifolds [6]. The Selberg's trace formula for hyperbolic groups (no points in H^2 fixed under the action of the group) in the two-dimensional hyperboloid H^2 is given in [7] and describe the spectra of oriented two-dimensional manifolds without boundaries. For bordered hyperbolic spaces the Selberg's trace formula is given in [8, 9].

The purpose of this paper is to establish the dependence of the Casimir energy on the boundary configuration for M-polygons X in H^2 (polygons with M vertices). Using the Selberg's trace formula and the point splitting regularization method, we calculate the Casimir energy for a minimally coupled massless scalar field. The explicit formula which defines the dependence of the Casimir energy on the number of vertices is established. Namely, at $M \to \infty$ we have

$$E = \frac{C_0}{R} M \ln M, \tag{1}$$

where C_0 is the positive number given by (70).

This paper is arranged as follows. To establish notations in section 2 we give the short derivation of the Selberg's trace formula for hyperbolic polygons. Section 3 is devoted to the calculation of the Casimir energy for a massless scalar field in M-polygons. In the appendix, we give an integral transformation which we use to calculate the Selberg's trace formula.

2. The Selberg's trace formula for hyperbolic polygons

In this section, we give the short derivation of the Selberg's trace formula for hyperbolic polygons. This formula is a special case of the Selberg's trace formula for bordered Riemannian spaces [9].

The two-dimensional hyperboloid H^2 in the Lobachevski realization is the upper half plane $\mathrm{Im}(z) > 0$ equipped with the metric

$$\mathrm{d}s^2 = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}, \qquad z = x + \mathrm{i}y. \tag{2}$$

Geodesics in H^2 are circles centered at the real axis and lines parallel to the imaginary axis. We may parameterize a geodesic by two points on the extended real line which is the boundary of H^2. By (x, x') we denote the circle centered at the real line and passing through the points x and x'. By (x, ∞) we denote the line parallel to the imaginary axis and passing through the point x on the real line.

Consider a polygon X between M geodesics $L_1, L_2, \ldots, L_M$. Let q_j be the reflection with respect to the geodesic L_j and Γ be the group generated by the reflections $q_1, q_2, \ldots, q_M$. We require that X is the fundamental domain of the discrete group Γ.

From the Green function

$$G_\rho(z, w) = -\frac{1}{2\pi\sqrt{2}} \int_d^\infty \mathrm{d}y \frac{\mathrm{e}^{-\mathrm{i}\rho y}}{\sqrt{\cosh y - \cosh d}} \tag{3}$$

on the two-dimensional hyperboloid [10], we construct the one on X which satisfies the Dirichlet boundary conditions. Here, $d = d(z, w)$ is the invariant distance given by

$$\cosh d(z, w) = 1 + \frac{|z - w|^2}{2\,\mathrm{Im}\, z\, \mathrm{Im}\, w}. \tag{4}$$

The desired Green function is

$$\bar{G}_\rho(z, w) = \sum_{\gamma\in\Gamma} \chi(\gamma) G_\rho(\gamma z, w), \tag{5}$$

where

$$\chi(q_j) = -1 \tag{6}$$

is the one-dimensional representation of Γ. It satisfies the equation

$$\left[L + \rho^2 + \tfrac{1}{4}\right]\bar{G}_\rho(z, w) = y^2\delta(x - x')\delta(y - y'), \tag{7}$$

where

$$L = y^2\left(\partial_x^2 + \partial_y^2\right) \tag{8}$$

is the Laplace–Beltrami operator on H^2. From (5) we observe the property

$$\bar{G}_\rho(q_j z, w) = -\bar{G}_\rho(z, w). \tag{9}$$

On the other side, the reflection q_j leaves the geodesic L_j fixed. Thus, the Green function vanishes on the boundary of X.

Consider a two-point function

$$\bar{K}(z, w) = \frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty} \mathrm{d}\rho\, \rho h(\rho)\bar{G}_\rho(z, w), \tag{10}$$

where $h(\rho)$ is a function such that the integral is well defined. The spectral representation

$$\bar{G}_\rho(z, w) = \sum_{n=0}^{\infty} \frac{\Psi_n(z)\overline{\Psi_n(w)}}{\rho^2 - \rho_n^2} \tag{11}$$

for the Green function gives the trace formula

$$Z[h] = \sum_{n=0}^{\infty} h(\rho_n) = \int_X \mathrm{d}\mu\, \bar{K}(w, w). \tag{12}$$

On the other side, the geodesic representation (5) for the Green function implies

$$Z[h] = \sum_{\gamma\in\Gamma} \chi(\gamma)\int_X \mathrm{d}\mu\, K(\gamma w, w). \tag{13}$$

The Selberg's trace formula manifests the equality between the sum over spectra and the sum over geodesics. Here, $\left\{\rho_n^2 + \frac{1}{4}\right\}_{n=0}^{\infty}$ and $\{\Psi_k(\vec{x})\}_{n=0}^{\infty}$ are the set of eigenvalues and eigenfunctions, respectively, of the Laplacian $-L$,

$$\mathrm{d}\mu = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \tag{14}$$

is the invariant measure on the hyperboloid H^2 and

$$K(z, w) = \frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty} \mathrm{d}\rho\, \rho h(\rho) G_\rho(z, w), \tag{15}$$

with $G_\rho(z, w)$ being the free Green function (5).

Let H be the set of equivalent classes in Γ defined by the relation: one identifies γ and γ' if there exist $\tilde{\gamma} \in \Gamma$ such that

$$\gamma' = \tilde{\gamma}\gamma\tilde{\gamma}^{-1}. \tag{16}$$

The summation over Γ can be replaced by the summation over the conjugacy classes H

$$Z[h] = \sum_{\gamma\in H} \chi(\gamma) \int_{H^2/\Sigma_\gamma} \mathrm{d}\mu\, K(\gamma w, w), \tag{17}$$

where

$$\Sigma_\gamma = \{\tilde{\gamma} \in \Gamma : \gamma\tilde{\gamma} = \tilde{\gamma}\gamma\} \tag{18}$$

is the stability group of $\gamma \in H$. The construction of the trace formula is reduced to the problem of the classification of conjugacy classes and their stability groups.

The stability group of the unit element is the whole group Γ and the contribution to the trace formula is [11]

$$Z_0[h] = \frac{S}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\rho\, \rho \tanh \pi\rho h(\rho), \tag{19}$$

where S is the area of X.

Let the intersection point V_j of geodesics L_j and L_{j+1} has the period P_j, that is the angle between these geodesics is $\frac{2\pi}{P_j}$. We assume that P_j is an even number. The reflections q_j and q_{j+1} generate a finite group G which leaves the vertex V_j fixed. Every geodesic in H^2 can be obtained from the imaginary axis

$$L_0 = (0, \infty) \tag{20}$$

by a conformal transformation

$$\gamma z = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \tag{21}$$

where a, b, c, d are real numbers. By transforming the geodesic L_j into L_0 and the vertex V_j into the point $z = \mathrm{i}$, the reflections q_j and q_{j+1} take the following form:

$$q_j = q_0, \qquad q_{j+1} = kq_0, \tag{22}$$

where

$$q_0 z = -\bar{z} \tag{23}$$

is the reflection with respect to the geodesic L_0 and

$$k = \begin{pmatrix} \cos\frac{2\pi}{P_j} & \sin\frac{2\pi}{P_j} \\ -\sin\frac{2\pi}{P_j} & \cos\frac{2\pi}{P_j} \end{pmatrix} \tag{24}$$

is the rotation around the point $z = \mathrm{i}$ of the order $\frac{P_j}{2}$. Note that since we are dealing with projective transformations we have $k^{\frac{P_j}{2}} = -1 \equiv 1$. G is the dihedral group of the order P_j. The conjugacy classes in G are

$$\{q_0, kq_0\} \tag{25}$$

and

$$\left\{k, k^2, \ldots, k^{\frac{P_j}{4}}\right\} \tag{26}$$

if $\frac{P_j}{2}$ is even and

$$\left\{k, k^2, \ldots, k^{\frac{P_j-2}{4}}\right\} \tag{27}$$

if $\frac{P_j}{2}$ is odd. For the latter case, Σ_{k^m} is the cyclic subgroup G_0 of G generated by k. For the even case, we have $\Sigma_{k^m} = G_0$ if $m \neq \frac{P_j}{4}$ and $\Sigma_{k^m} = G$ if $m = \frac{P_j}{4}$. The contribution from the vertex V_j to the trace formula is (see (A.14))

$$Z_{V_j}[h] = \frac{1}{P_j} \sum_{n=1}^{\frac{P_j}{2}-1} \int_0^\infty \frac{\mathrm{d}y\, g(y) \cosh y}{\sinh^2 y + \sin^2 \frac{2\pi n}{P_j}}, \tag{28}$$

where

$$g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\rho\, h(\rho)\, \mathrm{e}^{-\mathrm{i}\rho y}. \tag{29}$$

The total contribution to $Z[h]$ from the vertices of X is

$$Z_V[h] = \sum_{j=1}^{M} Z_{V_j}(s). \tag{30}$$

Let Λ be the set of all geodesics on H^2 which can be obtained from the geodesics L_j, $j = 1, \ldots, M$ which form the boundary of X by the action of the group Γ. We consider a class of fundamental domains X with the following property: for any nonintersecting pair of geodesics L and L' in Λ there exist a geodesic $\tilde{L}$ in Λ which intersects L and L' at the right angles[1]. If q and q' are the reflections with respect to the geodesics L and L' then $\gamma = qq'$ is the translation along the geodesic $\tilde{L}$, that is γ transforms this geodesic into itself. Thus, for any geodesic in Λ there exists a translation along this geodesic.

Now we are ready to consider the reflection q_0 of (25). Σ_{q_0} is generated by the reflection q_0 and by the translation

$$\gamma = \begin{pmatrix} \mathrm{e}^{\frac{l_\gamma}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{l_\gamma}{2}} \end{pmatrix} \tag{31}$$

along the imaginary axis L_0. Here, l_γ is the length of γ. The fundamental domain Ω of Σ_{q_0} is the half of the strip between two circles of radii 1 and e^{l_γ} centered at $z = 0$:

$$\Omega = \{z \in H^2 : 1 < |z| < \mathrm{e}^{l_\gamma}, \mathrm{Re}\, z > 0\}. \tag{32}$$

The contribution to the trace formula from the geodesic L_0 is given by (A.22). In a similar fashion one can treat other edges of the polygonal X. The trace formula from the edges is

$$Z_L[h] = \frac{g(0)}{4} \sum_{j=1}^{M} l_{\gamma_j}, \tag{33}$$

where γ_j is the length of the translation along the geodesic L_j.

Now we consider hyperbolic elements in H. Let $\bar{H}$ be the set of elements in H which have no fixed points on H^2. We decompose $\bar{H}$ into an even H_+ and an odd H_- pieces defined by the condition

$$H_\pm = (\gamma \in \bar{H} : \chi(\gamma) = \pm 1\}. \tag{34}$$

Elements of H_+ are translations γ which are conformal transformations (21) with the property $a + d > 2$. For any translation γ, there exist a geodesic L in Λ which is preserved by this translation. The reflection q with respect to L commutes with γ. Thus, Σ_γ is generated

[1] It seems that this property is valid for any discrete group Γ generated by reflections and having a nonzero compact fundamental domain in H^2, but the proof for the general case appears to be difficult.

by γ and q with the fundamental domain (32). The contribution from even hyperbolic transformations to the trace formula is (see (A.18))

$$Z_+[h] = \frac{1}{4}\sum_{\gamma\in H_+}\frac{l_\gamma g(l_\gamma)}{\sinh\frac{l_\gamma}{2}}. \tag{35}$$

Translations γ^n for positive integers n have the same stability group Σ_γ. Let A_+ be the set of primite elements in H_+ (all elements in H_+ can be obtained by multiplications of elements in A_+). The even trace formula can be rewritten as

$$Z_+[h] = \frac{1}{4}\sum_{\gamma\in A_+}\sum_{n=1}^{\infty}\frac{l_\gamma g(nl_\gamma)}{\sinh\frac{nl_\gamma}{2}}. \tag{36}$$

Let now $\gamma \in H_-$. A transformation γ^2 is a translation. Therefore, there exist a geodesic L which is preserved by it. Let q be the reflection with respect to L and $\gamma = q\gamma_0$. If we transport L into the imaginary axis L_0, the translation γ^2 will be of the diagonal form (31) with the length l_{γ^2} and the reflection q transforms into q_0 given by (23). Let

$$\gamma_0 = \begin{pmatrix} a & b\\ c & d\end{pmatrix}. \tag{37}$$

The action of q_0 in the matrix form is

$$q_0\begin{pmatrix} a & b\\ c & d\end{pmatrix}q_0 = \begin{pmatrix} a & -b\\ -c & d\end{pmatrix}. \tag{38}$$

The condition

$$\gamma^2 = \begin{pmatrix} \mathrm{e}^{\frac{l_{\gamma^2}}{2}} & 0\\ 0 & \mathrm{e}^{-\frac{l_{\gamma^2}}{2}}\end{pmatrix} \tag{39}$$

implies that γ_0 is of the diagonal form with $l_{\gamma_0} = \frac{1}{2}l_{\gamma^2}$. Thus, Σ_γ is generated by γ_0 and q with the fundamental domain (32) where now l_γ is defined to be the half of the length of the translation γ^2. The contribution from odd hyperbolic transformations to the trace formula is (see (A.21))

$$Z_-[h] = \frac{1}{4}\sum_{\gamma\in H_-}\frac{l_\gamma g(l_\gamma)}{\cosh\frac{l_\gamma}{2}}. \tag{40}$$

Transformations γ^{2n+1} for positive integers n have the same stability group. The odd trace formula takes the form

$$Z_-[h] = \frac{1}{4}\sum_{\gamma\in A_-}\sum_{n=1}^{\infty}\frac{l_\gamma g((2n+1)l_\gamma)}{\cosh\left(n+\frac{1}{2}\right)l_\gamma}, \tag{41}$$

where A_- is the set of primite elements in H_-.

Collecting all the above terms, we arrive at the Selberg's trace formula in X,

$$Z[h] = Z_0[h] + Z_V[h] - Z_L[h] + Z_+[h] - Z_-[h], \tag{42}$$

where $Z_0[h]$ given by (19) is the trace over the spectra of the two-dimensional hyperboloid H^2; $Z_V[h]$ and $Z_L[h]$ given by (30) and (33) include the effects of vertices and edges which form the boundary of X; $Z_+[h]$ and $Z_-[h]$ given by (36) and (41) are contributions from even and odd hyperbolic transformations. The Selberg's trace formula for hyperbolic groups which describe the spectra of oriented manifolds without boundaries contains the free Z_0 and even Z_+ terms only [7]. The remaining terms in (42) are related to the boundary effects.

Before closing this section, we give the example of symmetric M-polygon X with equal edges and angles between them. It is bounded by M geodesics

$$L_j = \tan\left(\frac{\phi}{2} + (j-1)\frac{\pi}{M}\right), \qquad \tan\left(-\frac{\phi}{2} + (j-1)\frac{\pi}{M}\right), \qquad j = 1, 2, \ldots, M, \tag{43}$$

where

$$\sin\phi = \frac{\sin\frac{\pi}{M}}{\cos\frac{\pi}{P}} \tag{44}$$

and P is the period of the vertices in X. The reflection with respect to L_j is

$$q_j = k^{j-1}\gamma q k^{-(j-1)}, \tag{45}$$

where γ is the diagonal matrix (31) with the length given by

$$\cosh l = \frac{\cos\frac{\pi}{P}}{\sin\frac{\pi}{M}}, \tag{46}$$

q is the reflection with respect to the geodesic $(-1, 1)$

$$qz = \frac{1}{\bar{z}} \tag{47}$$

and k is the rotation of the order M. When $M \gg 1$, the length of the translation γ becomes much greater than 1. For symmetric M-polygon X with sufficiently large number of vertices, we can neglect the even $Z_+[h]$ and the odd $Z_-[h]$ terms in the Selberg's trace formula. As a result, we have

$$Z[h] \simeq Z_0[h] + Z_V[h] + Z_L[h], \qquad M \gg 1. \tag{48}$$

For $P = 4$, that is when the angles between geodesics are right ones, we have

$$Z[h] \simeq \frac{S}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\rho\, \rho \tanh \pi\rho h(\rho) + \frac{M}{4}\int_0^{\infty} \frac{\mathrm{d}y\, g(y)}{\cosh y} + \frac{M\ln M}{4} g(0). \tag{49}$$

If M is even then one may glue the opposite edges of X to get the sphere with $g = \frac{M}{2}$ handles. By the Gauss–Bonnet theorem, the area of X is $4\pi(g-1)$.

3. The Casimir energy in polygons

We consider a scalar field in the three-dimensional Clifford–Klein spacetime

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2\frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}. \tag{50}$$

By quantizing canonically the scalar field in a polygon X, the two-point function is given by $(t > t')$ formal expression

$$\langle 0|\phi(x)\phi(x')|0\rangle = \sum_k \frac{\mathrm{e}^{-\mathrm{i}\omega_k(t-t'-\mathrm{i}\varepsilon)}}{2\omega_k}\Psi_k(\vec{x})\Psi_k(\vec{x}'), \tag{51}$$

where

$$\omega_n^2 = \frac{1}{R^2}\left(\rho_n^2 + \frac{1}{4}\right) \tag{52}$$

and $\left\{\rho_n^2 + \frac{1}{4}\right\}_{n=0}^{\infty}$ and $\{\Psi_k(\vec{x})\}_{n=0}^{\infty}$ are the set of eigenvalues and eigenfunctions, respectively, of the Laplacian $-L$. As long as ε is kept different from zero, the two-point function is a well-defined quantity, which is (for $t > t'$) related to the positive frequency part of the

Feynman propagator. The vacuum energy density $E(\vec{x})$ can be obtained from this propagator by applying a bi-differential operator and then by taking a coincidence limit [12]

$$E(\vec{x}) = \frac{1}{2} \lim_{(t',x',y')\to(t,x,y)} [\partial_t \partial_{t'} + y^2(\partial_x \partial_{x'} + \partial_y \partial_{y'})] \langle 0|\phi(x)\phi(x')|0\rangle. \tag{53}$$

Integrating the energy density over X, we arrive at

$$E_\varepsilon = -\frac{1}{2R}\frac{\partial}{\partial\varepsilon}\sum_{n=0}^{\infty} \mathrm{e}^{-\varepsilon\sqrt{\rho_n^2+\frac{1}{4}}}, \tag{54}$$

which is the regularization in which we are interested: the Casimir energy is a finite part of E_ε in $\varepsilon \to 0$ limit. For static spacetimes, this regularization is equivalent to another regularization techniques [13].

The Selberg's trace formula (42) gives the possibility to calculate the Casimir energy in an arbitrary M-polygon X

$$E_\varepsilon = -\frac{1}{2R}\frac{\partial}{\partial\varepsilon} Z[h], \tag{55}$$

where

$$h(\rho) = \mathrm{e}^{-\varepsilon\sqrt{\rho^2+\frac{1}{4}}}. \tag{56}$$

To establish the dependence of the vacuum energy in X on the boundary configuration explicitly, we consider the symmetric M-polygon with sufficiently large number of vertices which we introduced in the previous section. For the sake of simplicity, we consider the case $P = 4$ of (49). The Selberg's trace formula (49) implies

$$E \simeq E_0 + E_V + E_L. \tag{57}$$

The bulk energy of the Clifford–Klein spacetime

$$E_0 = -\frac{1}{2R}\frac{\partial}{\partial\varepsilon} Z_0[h] \tag{58}$$

is negative [14]:

$$E_0 = -\frac{S}{2\pi R}\int_0^\infty \rho\, \mathrm{d}\rho \frac{\sqrt{\rho^2+\frac{1}{4}}}{1+\mathrm{e}^{2\pi\rho}} - \frac{S}{96\pi R}, \tag{59}$$

where S is the area of X.

The effect of vertices is given by

$$E_V = -\frac{1}{2R}\frac{\partial}{\partial\varepsilon} Z_V[h] \tag{60}$$

or

$$E_V = -\frac{M}{8R}\int_0^\infty \frac{\mathrm{d}y}{\cosh y}\frac{\partial}{\partial\varepsilon} g_\varepsilon(y), \tag{61}$$

where

$$g_\varepsilon(y) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\rho \cos y\rho\, \mathrm{e}^{-\varepsilon\sqrt{\rho^2+\frac{1}{4}}}. \tag{62}$$

Using the Taylor expansion for $\cos y\rho$ and the integral representation

$$K_m\left(\frac{\varepsilon}{2}\right) = \frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}\varepsilon^m \int_0^\infty \mathrm{d}x\, x^{2m} \frac{\mathrm{e}^{-\varepsilon\sqrt{\rho^2+\frac{1}{4}}}}{\sqrt{\rho^2+\frac{1}{4}}} \tag{63}$$

for the modified Bessel function (see page 959 of [16]), we arrive at

$$g_\varepsilon(y) = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-)^m}{m!}\left(\frac{y}{2}\right)^{2m}\frac{\partial}{\partial\varepsilon}\frac{K_m\left(\frac{\varepsilon}{2}\right)}{\varepsilon^m}. \tag{64}$$

The integral (see page 349 of [16])

$$\int_0^\infty \frac{y^{2m}}{\cosh y} = \left(\frac{\pi}{2}\right)^{2m+1}|E_{2m}| \tag{65}$$

allows us to rewrite (61) as

$$E_V = \frac{M}{16R}\sum_{m=0}^{\infty}\frac{(-)^m|E_{2m}|}{m!}\left(\frac{\pi}{4}\right)^{2m}\frac{\partial^2}{\partial\varepsilon^2}\frac{K_m\left(\frac{\varepsilon}{2}\right)}{\varepsilon^m}. \tag{66}$$

Here, E_m are the Euler's numbers which can be defined from the expansion

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty}\frac{E_n}{n!}z^n, \qquad |z| < \frac{1}{2}. \tag{67}$$

From the series representation for the modified Bessel function (see page 961 of [16])

$$K_m(z) = \frac{1}{2}\sum_{k=0}^{m-1}(-)^m\frac{(m-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-m} + (-)^{m+1}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k+m}}{k!(m+k)!}\left[\ln\frac{z}{2} - \frac{\Psi(k+1)+\Psi(m+k+1)}{2}\right], \tag{68}$$

we observe that only the second term at $k = 1$ has a finite part in $\varepsilon \to 0$ limit:

$$\frac{\partial^2}{\partial\varepsilon^2}\frac{K_m\left(\frac{\varepsilon}{2}\right)}{\varepsilon^m} = (-)^{m+1}\frac{C_m}{4^{m+2}(m+1)!}, \tag{69}$$

where

$$C_m = 2 + 2C + 4\ln 2 - \sum_{k=1}^{m+1}\frac{1}{k} \tag{70}$$

and $C = 0.577\ldots$ is the Euler number. The Casimir effect due to the vertices is

$$E_V = -\frac{M}{256R}\sum_{m=0}^{\infty}\frac{|E_{2m}|}{m!(m+1)!}\left(\frac{\pi}{8}\right)^{2m}C_m. \tag{71}$$

From equation (68), we observe that for $\varepsilon \to 0$ there exists logarithmic divergence. The corresponding subtraction therefore depends on a scale of the polygon (i.e., its perimeter). Since we are dealing with a regular polygon this scale does not appear in the trace formula explicitly. If we reinstitute a length scale, the log is not simply of the number of vertices, but rather of dimensionful length, and must be compared to some other length scale. The obvious one is the curvature, but the subtraction in the vacuum energy of the polygon in principle is ambiguous by a term proportional to M, i.e., proportional to the number of vertices. This (I believe) is not terrible, since whereas areas and edge length can be changed adiabatically to an assemble of polygons, the number of vertices cannot. A contribution to the vacuum energy of a polygon proportional to the number of vertices therefore is not observable.

The effect of edges which form the boundary of X is given by

$$E_L = \frac{1}{2R}\frac{\partial}{\partial\varepsilon}Z_L[h] \tag{72}$$

or

$$E_L = \frac{M \ln M}{8R} \int_0^\infty \frac{\mathrm{d}y}{\cosh y} \frac{\partial}{\partial \varepsilon} g_\varepsilon(0). \tag{73}$$

The integral representation (63) implies

$$E_L = -\frac{M \ln M}{8\pi R} \int_0^\infty \frac{\mathrm{d}y}{\cosh y} \frac{\partial^2}{\partial \varepsilon^2} K_0\left(\frac{\varepsilon}{2}\right). \tag{74}$$

By the virtue of (69) we get the final result

$$E_L = \frac{C_0}{128\pi R} M \ln M, \tag{75}$$

which is positive. At $M \to \infty$, the contribution to the Casimir energy from the edges becomes much greater than the bulk energy E_0:

$$E \simeq \frac{C_0}{128\pi R} M \ln M. \tag{76}$$

The Casimir energy for the symmetric M-polygon with sufficiently large number of vertices is positive and increases with the number of vertices. To see the rate of the increasing, we divide the above expression on the area of X which is for even M is given by $S = 2\pi M$ (when $M \gg 1$, the area for any polygon X is a linear function of M). The vacuum energy per unit area increases as the logarithm of M.

Before closing this section, we shortly discuss the contribution to the vacuum energy from the even Z_+ and the odd Z_- terms in the Selberg's trace formula. The Casimir effect from even hyperbolic transformations in Γ is

$$E_+ = -\frac{1}{32\pi R} \sum_{\gamma \in A_+} \sum_{n=1}^{\infty} \frac{K_1\left(\frac{n l_\gamma}{2}\right)}{n \sinh \frac{n l_\gamma}{2}} \tag{77}$$

which appears to be negative, as E_0 and E_V. The latter ones are also related to even transformations in Γ (transformations which do not contain a reflection). Even elements in the fundamental group give negative contribution to the vacuum energy. If Γ is strictly hyperbolic group then we have only even transformations. As a result, the Casimir energy for oriented two-dimensional hyperbolic manifolds without boundaries is negative [14, 15].

On the other side, the Casimir effect from odd hyperbolic transformations

$$E_- = \frac{1}{32\pi R} \sum_{\gamma \in A_-} \sum_{n=0}^{\infty} \frac{K_1\left(\frac{(2n+1) l_\gamma}{2}\right)}{(2n+1) \cosh \frac{(2n+1) l_\gamma}{2}} \tag{78}$$

is positive. The vacuum energy E_L from edges of X is also due to odd transformations in Γ: odd transformations give positive contribution to the vacuum energy.

Before closing this section, we shortly discuss the relation between the group Γ_X which defines X and the spectrum Π_X of the Laplacian (8) on X. Let us assume that there exist some geodesic in X such that the reflection with respect to it is the symmetry transformation of X. Let X' be the subspace of X which is obtained by the identification of opposite points in X. The fundamental group of X' is 'bigger' than the one of X: $\Gamma_{X'}$ is generated by elements of Γ_X and by the above reflection. On the other side, the spectrum $\Pi_{X'}$ of X' is 'smaller' than the one of X: one has to impose addition conditions on Π_X to arrive at $\Pi_{X'}$.[2] We have the following duality

$$\Gamma_X \subset \Gamma_{X'} \quad \Longrightarrow \quad \Pi_{X'} \subset \Pi_X \tag{79}$$

[2] For the sake of simplicity, consider the flat case. The spectra in a unit square X is $\Sigma_X = \{p = \pi(n, m), n, m = 1, 2, \ldots\}$. By the reflection with respect to the diagonal, we arrive at the right square X' with $\Sigma'_X = \{p = \pi(n, m), n < m = 2, 3, \ldots\}$ which is the subset of Σ_X.

between transformations which defines polygons in H^2 and the spectra of the Laplacian on these polygons. We have shown that the even and odd transformations of the fundamental group Γ give negative and positive contributions to the vacuum energy, respectively. The above duality between group transformations and the spectra of the Laplace–Beltrami operator may shed light on the origin of the Casimir energy sign: one may find those modes in the spectra which are responsible for increasing or decreasing of the Casimir energy. However, since no explicit formulae for the spectra Σ_X and the duality (79) are known for hyperbolic polygons this question appears to be open.

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Appendix

Function (15) can be rewritten as

$$K(z,w) = -\frac{1}{\pi\sqrt{2}}\int_d^\infty \mathrm{d}y\,\frac{g'(y)}{\sqrt{\cosh y - \cosh d}} \tag{A.1}$$

or

$$\Phi(\xi) = -\frac{1}{\pi}\int_\xi^\infty \frac{\mathrm{d}\eta\, Q'(\eta)}{\sqrt{\eta-\xi}}, \tag{A.2}$$

where we introduced new variables

$$\Phi(2(\cosh d - 1)) = K(z,w) \tag{A.3}$$

and

$$Q(2(\cosh y - 1)) = g(y). \tag{A.4}$$

Here, d is the geodesic distance (4) and $g(y)$ is the function given by (29).

Integral (A.2) has the inverse

$$Q(\eta) = \int_{-\infty}^\infty \mathrm{d}\xi\, \Phi(\eta+\xi^2). \tag{A.5}$$

In the new variable $\tau^2 = \eta - \xi$, expression (A.2) reads

$$\Phi(\xi) = -\frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\xi}Q(\xi+\tau^2). \tag{A.6}$$

Using (A.5) we have

$$\Phi(\xi) = -\frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\tau\,\mathrm{d}\sigma\,\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi(\xi+\tau^2+\sigma^2), \tag{A.7}$$

which in polar coordinates reads

$$\Phi(\xi) = -2\int_0^\infty r\,\mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi(\xi+r^2) = -\int_0^\infty \mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\Phi(\xi+x) = \Phi(\xi). \tag{A.8}$$

Thus, we have shown that (A.5) is the inverse transform of (A.2).

(a) Let $\frac{P}{2}$ be an odd number. Then the contribution from k^n to the trace formula is

$$I(k^n) = \int_{H^2/G_0} \mathrm{d}\mu\, K(k^n w, w) = \frac{2}{P}\int_{H^2}\mathrm{d}\mu\, K(k^n w, w), \tag{A.9}$$

where P is the period of a vertex and G_0 is the rotational group of the order $\frac{P}{2}$. The distance is

$$\cosh d(k^n w, w) = 1 + \frac{\sin^2 \frac{2\pi n}{P}}{2y^2}|1+w^2|^2. \tag{A.10}$$

In the polar coordinates $w = \mathrm{e}^{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}$, we have

$$I(k^n) = \frac{2}{P}\int_{-\infty}^{\infty}\mathrm{d}\alpha \int_{-\infty}^{\infty}\mathrm{d}z\, \Phi\left(4\sin^2\frac{2\pi n}{P}(\sinh^2\alpha + z^2\cosh^2\alpha)\right), \tag{A.11}$$

where $z = \cot\psi$. By the virtue of (A.5) we get

$$I(k^n) = \frac{1}{P\left|\sin\frac{2\pi n}{P}\right|}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{\cosh\alpha} Q\left(4\sin^2\frac{2\pi n}{P}\sinh^2\alpha\right) \tag{A.12}$$

or

$$I(k^n) = \frac{1}{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sin^2\frac{2\pi n}{P} + t^2} Q(4t^2), \tag{A.13}$$

where $t = \sin\frac{2\pi n}{P}\sinh\alpha$. If we put $t = \sinh y$, then the above formula together with (A.4) implies

$$I(k^n) = \frac{1}{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}y\, g(y)\cosh y}{\sin^2\frac{2\pi n}{P} + \sinh^2 y}. \tag{A.14}$$

(b) The contribution from $\gamma \in H_+$ to the trace formula is

$$I(\gamma) = \frac{1}{2}\int_{1<|z|<\mathrm{e}^{l_\gamma}}\mathrm{d}\mu\, K(\gamma w, w), \tag{A.15}$$

where γ is the translation (31) with the length l_γ. The distance is

$$\cosh d(\gamma w, w) = 1 + 2\sinh^2\frac{l_\gamma}{2}\frac{|w|^2}{y^2}. \tag{A.16}$$

In the polar coordinates we have

$$I(\gamma) = \frac{1}{2}\int_1^{\mathrm{e}^{l_\gamma}}\frac{\mathrm{d}r}{r}\int_{-\infty}^{\infty}\mathrm{d}z\,\Phi\left(4\sinh^2\frac{l_\gamma}{2}(1+z^2)\right) \tag{A.17}$$

or

$$I(\gamma) = \frac{l_\gamma}{4\sinh\frac{l_\gamma}{2}} g\left(\frac{l_\gamma}{2}\right). \tag{A.18}$$

(c) The contribution from $\gamma \in H_-$ to the trace formula is

$$J(\gamma) = \frac{1}{2}\int_{1<|z|<\mathrm{e}^{l_\gamma}}\mathrm{d}\mu\, K(\gamma w, w). \tag{A.19}$$

Here, $\gamma = q_0\gamma_0$ with q_0 and γ_0 being the reflection (23) and the translation (31) with the length l_γ. In the polar coordinates, we have

$$d(q_0\gamma_0 w, w) = 1 + 2\left(\sinh^2\frac{l_\gamma}{2} + \cosh^2\frac{l_\gamma}{2}z^2\right). \tag{A.20}$$

Formula (A.5) implies

$$J(\gamma) = \frac{l_\gamma g\left(\frac{l_\gamma}{2}\right)}{4\cosh\frac{l_\gamma}{2}}. \tag{A.21}$$

For $\gamma_0 = 1$ we have

$$J(q_0) = \frac{l_\gamma g(0)}{4}. \tag{A.22}$$

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